The Combinatorics of the Euro

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Beginning in January 2002, twelve European countries will switch over to the *euro* currency. The following summarizes some interesting combinatorial facts about the coin/change structure of this new currency.

Standard US currency has coins of value (in cents): 1, 5, 10, 25, 50, 100.
The new euro currency has coins of value (in cents): 1, 2, 5, 10, 20, 50, 100, 200.

The following table compares the number of possible ways to make n cents (¢) change using US coins ($) or euro coins (€):

¢	$	€	¢	$	€	¢	$	€	¢	$	€	¢	$	€
1	1	1	21	9	44	41	31	249	61	77	828	81	159	2141
2	1	2	22	9	51	42	31	271	62	77	881	82	159	2249
3	1	2	23	9	54	43	31	284	63	77	916	83	159	2326
4	1	3	24	9	61	44	31	306	64	77	969	84	159	2434
5	2	4	25	13	68	45	39	328	65	93	1022	85	187	2542
6	2	5	26	13	75	46	39	350	66	93	1075	86	187	2650
7	2	6	27	13	82	47	39	372	67	93	1128	87	187	2758
8	2	7	28	13	89	48	39	394	68	93	1181	88	187	2866
9	2	8	29	13	96	49	39	416	69	93	1234	89	187	2974
10	4	11	30	18	109	50	50	451	70	112	1311	90	218	3121
11	4	12	31	18	116	51	50	473	71	112	1364	91	218	3229
12	4	15	32	18	129	52	50	508	72	112	1441	92	218	3376
13	4	16	33	18	136	53	50	530	73	112	1494	93	218	3484
14	4	19	34	18	149	54	50	565	74	112	1571	94	218	3631
15	6	22	35	24	162	55	62	600	75	134	1648	95	252	3778
16	6	25	36	24	175	56	62	635	76	134	1725	96	252	3925
17	6	28	37	24	188	57	62	670	77	134	1802	97	252	4072
18	6	31	38	24	201	58	62	705	78	134	1879	98	252	4219
19	6	34	39	24	214	59	62	740	79	134	1956	99	252	4366
20	9	41	40	31	236	60	77	793	80	159	2064	100	293	4563

These values can be computed as follows: Given coins of values $v_1, v_2, \ldots, v_k$, if we consider the product

$$(1+x^{v_1}+x^{2v_1}+\cdots)(1+x^{v_2}+x^{2v_2}+\cdots)\cdots(1+x^{v_k}+x^{2v_k}+\cdots) = \alpha_0+\alpha_1 x+\alpha_2 x^2+\alpha_3 x^3+\cdots$$

then the coefficient α_i of x^i is equal to the number of ways that i can be expressed as the sum of v_i's. This is because every cross-term in the product corresponds to exactly one way of giving change with coins of value $v_1, v_2, \ldots, v_k$, and the exponent of x represents the amount of change that is made.

Since $1/(1-x^n) = 1+x^n+x^{2n}+\cdots$, the above computation amounts to computing the power-series expansion (i.e. Taylor series) of

$$(1+x^{v_1}+x^{2v_1}+\cdots)(1+x^{v_2}+x^{2v_2}+\cdots)\cdots(1+x^{v_k}+x^{2v_k}+\cdots) = \frac{1}{(1-x^{v_1})(1-x^{v_2})\cdots(1-x^{v_k})}$$

which is easily done by a symbolic computation package such as Mathematica. For instance, in the case of euros, we have:

$$\frac{1}{(1-x^1)(1-x^2)(1-x^5)(1-x^{10})(1-x^{20})(1-x^{50})(1-x^{100})} = 1+x+2x^2+2x^3+3x^4+4x^5+\cdots$$

Another measure of how flexible the euro coins are compared to US currency is the following: "What is the minimal set of coins that will allow one to create change for any value from 1 to 100 cents?"

8 suffice using euros: 1×1 cent, 2×2 cent, 1×5 cent, 2×10 cent, 1×20 cent, 1×50 cent

9 are necessary for US change: 4×1 cent, 1×5 cent, 2×10 cent, 1×25 cent, 1×50 cent

Note: Since, given k coins, there are 2^k possible subsets of those coins, the minimum possible such set of coins (for coins of any values) is equal to the next integer larger than $\log_2 100 = 6.6438\ldots$, namely 7.

Let $f_{€}(n)$ and $f_{\$}(n)$ denote the minimum number of coins necessary to make n cents of change using euros and US currency respectively:

The average value of $f_{€}(n)$ as n ranges from 1 to 100 is 3.41.

The average value of $f_{\$}(n)$ as n ranges from 1 to 100 is 4.21.